



# A vector asymmetrical NNV equation: Soliton solutions, bilinear Bäcklund transformation and Lax pair

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## Abstract

A vector asymmetrical Nizhnik–Novikov–Veselov (NNV) equation is proposed based on its bilinear form. Soliton solutions expressed by Pfaffians are obtained. Bilinear Bäcklund transformation and the corresponding Lax pair for the vector ANNV equation are derived.

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## 1. Introduction

In the literature, several approaches have been developed to search for integrable coupled systems, such as the celebrated Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0. \quad (1)$$

One of them is the Hirota's bilinear method. We first recall how to deduce coupled KdV system by the bilinear formalism. It is known that the KdV equation (1) can be transformed into the bilinear form

$$D_x(D_t + D_x^3)f \cdot f = 0 \quad (2)$$

by the dependent variable transformation

$$u = 2(\ln f)_{xx}, \quad (3)$$

where the bilinear operators  $D_x^m D_t^k$  are defined by [1]

$$D_y^m D_t^k a \cdot b \equiv \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^k a(y, t) b(y', t') \Big|_{y'=y, t'=t}.$$

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Based on the fact that the bilinear KdV equation (2) can be rewritten as  $(D_t + D_x^3)f_x \cdot f = 0$ , a new bilinear form for the KdV equation (1) is given as [2]

$$(D_t + D_x^3)g \cdot f = 0, \quad (4)$$

$$D_x^2 f \cdot f = 2D_x g \cdot f, \quad (5)$$

which has a natural coupled form

$$(D_t + D_x^3)g_j \cdot f = 0, \quad j = 1, 2, \dots, M, \quad (6)$$

$$D_x^2 f \cdot f = 2D_x \left( \sum_{j=1}^M g_j \right) \cdot f. \quad (7)$$

By the dependent variable transformation  $v_j = 2g_j/f$ , Eqs. (6) and (7) can be transformed into an  $M$ -component potential KdV equation

$$\frac{\partial v_j}{\partial t} + 3 \left( \sum_{k=1}^M \frac{\partial v_k}{\partial x} \right) \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0, \quad j = 1, 2, \dots, M, \quad (8)$$

or its vector form

$$\mathbf{v}_t + 3(\mathbf{c} \cdot \mathbf{v})_x \mathbf{v}_x + \mathbf{v}_{xxx} = 0, \quad (9)$$

where  $\mathbf{v} = (v_1, v_2, \dots, v_M)$ ,  $\mathbf{c} = (1, 1, \dots, 1)$  and the inner product  $\mathbf{c} \cdot \mathbf{v}$  is defined by

$$\mathbf{c} \cdot \mathbf{v} = \sum_{i=1}^M v_i.$$

Set  $u_i = v_{ix}$ ,  $\mathbf{u} = \mathbf{v}_x$ ;  $M$ -component KdV equation is obtained

$$\frac{\partial u_i}{\partial t} + 3 \left( \sum_{k=1}^M u_k \right) \frac{\partial u_i}{\partial x} + 3 \left( \sum_{k=1}^M \frac{\partial u_k}{\partial x} \right) u_i + \frac{\partial^3 u_i}{\partial x^3} = 0, \quad i = 1, 2, \dots, M, \quad (10)$$

or its equivalent vector form

$$\mathbf{u}_t + 3[(\mathbf{c} \cdot \mathbf{u})\mathbf{u}]_x + \mathbf{u}_{xxx} = 0. \quad (11)$$

Other coupled systems from KdV equation can be seen in [3,4]. The vector Ito equation [2] for the Ito equation [5] is also successfully obtained by using the bilinear approach.

The two coupled systems, vector KdV equation and vector Ito equation, are both  $(1+1)$ -dimensional systems, a natural idea is to extend the bilinear approach to high dimension and search for coupled systems. In this paper we consider the vector form for the asymmetric Nizhnik–Novikov–Veselov (NVV) equation

$$u_t + u_{xxx} + 3 \left[ u \left( \int u_x dy \right) \right]_x = 0, \quad (12)$$

or

$$u_t + u_{xxx} + 3[uv]_x = 0; \quad u_x = v_y. \quad (13)$$

The asymmetric Nizhnik–Novikov–Veselov (ANNV) equation (13) may be considered as a model for an incompressible fluid where  $u$  and  $v$  are the components of the (dimensionless) velocity [6]. The ANNV equation (12) or (13) has the bilinear form [7]

$$u = 2(\ln f)_{xy}, \quad v = 2(\ln f)_{xx}, \quad (14)$$

$$D_y(D_t + D_x^3)f \cdot f = 0. \quad (15)$$

The success of extension from Eq. (2) to the coupled form (6) and (7) motivates one to propose a natural coupled form

$$(D_t + D_x^3)g_j \cdot f = 0, \quad j = 1, 2, \dots, M, \quad (16)$$

$$D_y^2 f \cdot f = 2D_y \left( \sum_{j=1}^M g_j \right) \cdot f, \quad (17)$$

for the ANNV equation (15). Eqs. (16) and (17) may be rewritten in an equivalent form

$$(D_t + D_x^3)g_j \cdot f = 0, \quad j = 1, 2, \dots, M, \quad (18)$$

$$\sum_{j=1}^M g_j = f_y, \quad (19)$$

which becomes

$$v_{it} + v_{ixxx} + 3(uv_i)_x = 0, \quad (20)$$

$$u_y = \sum_{j=1}^M v_{jx}, \quad (21)$$

or its vector form

$$v_t + v_{xxx} + 3(uv)_x = 0, \quad (22)$$

$$u_y = (c \cdot v)_x, \quad (23)$$

by the dependent variable transformation

$$v_i = 2 \frac{D_x g_i \cdot f}{f^2}, \quad u = 2(\ln f)_{xx}. \quad (24)$$

We call the system (22)–(23) the vector ANNV equation.

The purpose of this paper is to study the vector ANNV equation using the bilinear approach. We will give soliton solutions, bilinear Bäcklund transformation for (16)–(17) and Lax pair for (22)–(23).

This paper is organized as follows. In Section 2, we give soliton solutions for the bilinear ANNV equation expressed by Pfaffians. In Section 3, soliton solutions expressed by Pfaffians are found for the coupled ANNV equations (16)–(17). In Section 4, we give a bilinear Bäcklund transformation (BT) for Eqs. (16)–(17). Furthermore, the corresponding Lax pair for Eqs. (22)–(23) is obtained from the bilinear BT. Finally, conclusion and discussions are given in Section 5.

## 2. Multisoliton solution for the ANNV equation by a Pfaffian expression

For symmetrical NNV equation, its DKP type Pfaffian solution and BKP type Pfaffian solution are given in [8]. In [9] solutions for NNV equation are constructed by the Moutard transformation. In the following, we would discuss multisoliton solutions for the ANNV equation and give a different Pfaffian expression from those in [8]. The bilinear ANNV equation

$$(D_y D_t + D_y D_x^3)f \cdot f = 0, \quad (25)$$

or equivalently

$$f \left( \frac{\partial^2 f}{\partial y \partial t} + \frac{\partial^4 f}{\partial y \partial x^3} \right) - \frac{\partial f}{\partial y} \left( \frac{\partial f}{\partial t} + \frac{\partial^3 f}{\partial x^3} \right) + 3 \left( \frac{\partial^2 f}{\partial y \partial x} \frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x} \frac{\partial^3 f}{\partial y \partial x^2} \right) = 0, \quad (26)$$

has 3-soliton solution expressed as

$$f = 1 + \exp[\eta_1] + \exp[\eta_2] + \exp[\eta_3] + A_{12} \exp[\eta_1 + \eta_2] + A_{13} \exp[\eta_1 + \eta_3] + A_{23} \exp[\eta_2 + \eta_3] \\ + A_{12} A_{13} A_{23} \exp[\eta_1 + \eta_2 + \eta_3], \quad (27)$$

with

$$\begin{aligned}\exp(\eta_j) &= \exp(p_j x + q_j y - p_j^3 t + \eta_{j,0}), \\ A_{ij} &= \frac{(q_i - q_j)(p_i - p_j)}{(q_j + q_i)(p_i + p_j)}, \quad \text{for } i, j = 1, 2, 3,\end{aligned}\quad (28)$$

where  $p_j$ ,  $q_j$  and  $\eta_{j,0}$  are free parameters.

These expressions suggest the existence of  $N$ -soliton Pfaffian solutions to Eq. (25). Suppose that the exact solution to Eq. (25) can be expressed as

$$f = \text{pf}(d_0, \beta_0, a_1, \dots, a_N, b_1, \dots, b_N), \quad (29)$$

with the entries defined by

$$\begin{aligned}\text{pf}(d_0, a_j) &= \exp(\eta_j), & \text{pf}(d_0, b_j) &= -1, & \text{pf}(d_0, \beta_0) &= 1, \\ \text{pf}(a_j, a_k) &= -a_{j,k} \exp(\eta_j + \eta_k), & \text{pf}(a_j, \beta_0) &= 0, \\ \text{pf}(b_j, b_k) &= b_{j,k}, & \text{pf}(b_j, \beta_0) &= 1, & \text{pf}(a_j, b_k) &= \delta_{j,k}, \quad \text{for } j, k = 1, 2, \dots, N,\end{aligned}\quad (30)$$

where

$$\begin{aligned}\exp(\eta_j) &= \exp(p_j x + q_j y - p_j^3 t + \eta_{j,0}), \\ a_{j,k} &= \frac{p_j - p_k}{p_j + p_k}, & b_{j,k} &= \frac{q_j - q_k}{q_j + q_k}, & \delta_{j,k} &= \begin{cases} 1, & \text{for } j = k, \\ 0, & \text{for } j \neq k. \end{cases}\end{aligned}\quad (31)$$

In the following we will prove that  $f$  given by (29) satisfies Eq. (25). We introduce new variables  $c$ ,  $d_i$  defined by

$$\begin{aligned}\text{pf}(c, a_j) &= 0, & \text{pf}(c, b_j) &= q_j, & \text{pf}(c, \beta_0) &= 0, & \text{pf}(c, d_0) &= 0, \\ \text{pf}(d_i, a_j) &= p_j^i e^{\eta_j}, & \text{pf}(d_i, b_j) &= 0, & \text{pf}(d_i, c) &= 0, \\ \text{pf}(d_i, d_0) &= 0, & \text{pf}(d_i, \beta_0) &= 0, & j &= 1, 2, \dots, N, & i &= 1, 2, 3.\end{aligned}\quad (32)$$

Using the method described in [10], we obtain the following differential formulae for  $f$ :

$$\begin{aligned}f &= \text{pf}(d_0, \beta_0, \bullet), & \frac{\partial f}{\partial x} &= -\text{pf}(d_0, d_1, \bullet), & \frac{\partial^2 f}{\partial x^2} &= -\text{pf}(d_0, d_2, \bullet), \\ \frac{\partial^3 f}{\partial x^3} &= -\text{pf}(d_0, d_3, \bullet) - (d_0, d_1, d_2, \beta_0, \bullet), & \frac{\partial f}{\partial t} &= \text{pf}(d_0, d_3, \bullet) - 2 \text{pf}(d_0, d_1, d_2, \beta_0, \bullet), \\ \frac{\partial f}{\partial y} &= -\text{pf}(d_0, c, \bullet), & \frac{\partial^2 f}{\partial x \partial y} &= \text{pf}(d_0, c, d_1, \beta_0, \bullet), & \frac{\partial^3 f}{\partial x^2 \partial y} &= \text{pf}(d_0, c, d_2, \beta_0, \bullet), \\ \frac{\partial^4 f}{\partial x^3 \partial y} &= \text{pf}(d_0, c, d_3, \beta_0, \bullet) + \text{pf}(d_0, d_1, d_2, c, \bullet), & \frac{\partial^2 f}{\partial t \partial y} &= \text{pf}(d_0, c, d_3, \beta_0, \bullet) - 2 \text{pf}(d_0, d_1, d_2, c, \bullet),\end{aligned}$$

where we have denoted  $\{a_1, a_2, \dots, a_N, b_1, \dots, b_N\}$  by  $\{\bullet\}$  for simplicity. Substituting these relations into Eq. (25), we find that the bilinear equation (25) has the following structure:

$$\begin{aligned}3 \text{pf}(d_0, d_1, d_2, c, \bullet) \text{pf}(d_0, \beta_0, \bullet) &- 3 \text{pf}(d_0, c, \bullet) \text{pf}(d_0, d_1, d_2, \bullet) \\ &+ 3 \text{pf}(d_0, d_1, c, \beta_0, \bullet) \text{pf}(d_0, d_2, \bullet) - 3 \text{pf}(d_0, d_2, c, \bullet) \text{pf}(d_0, d_1, \bullet),\end{aligned}\quad (33)$$

which vanishes because of the Pfaffian identity [11]:

$$\begin{aligned}\text{pf}(\alpha_1, \alpha_2, \alpha_3, 1, 2, \dots, 2n-1) \text{pf}(\alpha_4, 1, 2, \dots, 2n-1) \\ - \text{pf}(\alpha_1, 1, 2, \dots, 2n-1) \text{pf}(\alpha_2, \alpha_3, \alpha_4, 1, 2, \dots, 2n-1) \\ + \text{pf}(\alpha_2, 1, 2, \dots, 2n-1) \text{pf}(\alpha_1, \alpha_3, \alpha_4, 1, 2, \dots, 2n-1) \\ - \text{pf}(\alpha_3, 1, 2, \dots, 2n-1) \text{pf}(\alpha_1, \alpha_2, \alpha_4, 1, 2, \dots, 2n-1) = 0.\end{aligned}\quad (34)$$

Thus we have proved  $f$  given by (29) satisfies the bilinear ANNV equation (25).

### 3. $N$ -soliton solution for the coupled ANNV equation

We have transformed the vector ANNV equation into the bilinear form

$$(D_t + D_x^3)g_\mu \cdot f = 0, \quad \text{for } \mu = 1, 2, \dots, M, \quad (35)$$

$$D_y^2 f \cdot f = 2 \sum_{\mu=1}^M D_y g_\mu \cdot f. \quad (36)$$

We note that the second bilinear form (36) can be transformed into the linear equation

$$\frac{\partial f}{\partial y} = \sum_{\mu=1}^M g_\mu. \quad (37)$$

In the following, we would express  $N$ -soliton solution to Eqs. (35) and (37) by Pfaffians. In fact we find that

$$f = \text{pf}(d_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_0), \quad (38)$$

$$g_\mu = \text{pf}(d_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_\mu), \quad \text{for } \mu = 1, 2, \dots, M, \quad (39)$$

where the entries of the Pfaffians are defined as follows:

$$\text{pf}(d_0, a_j) = \exp(\eta_j), \quad \text{pf}(d_0, b_j) = -1, \quad \text{pf}(d_0, \beta_0) = 1, \quad (40)$$

$$\text{pf}(a_j, a_k) = -a_{jk} \exp(\eta_j + \eta_k), \quad \text{pf}(a_j, b_k) = \delta_{j,k}, \quad \text{pf}(a_j, \beta_0) = 0, \quad (41)$$

$$\text{pf}(b_j, b_k) = b_{j,k}, \quad \text{pf}(d_0, \beta_\mu) = 0, \quad \text{pf}(b_j, \beta_0) = 1, \quad (42)$$

$$\text{pf}(a_j, \beta_\mu) = 0, \quad \text{pf}(b_j, \beta_\mu) = c_\mu(j), \quad (43)$$

$$\text{for } j, k = 1, 2, \dots, N \text{ and } \mu = 1, 2, \dots, M, \quad (44)$$

where

$$\eta_j = p_j x + q_j y - p_j^3 t + \eta_j^0, \quad (45)$$

$$a_{j,k} = (p_j - p_k)/(p_j + p_k), \quad (46)$$

$$b_{j,k} = (q_j - q_k)/(q_j + q_k), \quad (47)$$

$$\delta_{j,k} = \begin{cases} 1, & \text{for } j = k, \\ 0, & \text{for } j \neq k. \end{cases} \quad (48)$$

$p_j, q_j$  and  $\eta_j^0$  are free parameters and  $c_\mu(j)$  are parameters satisfying a condition

$$\sum_{\mu=1}^M c_\mu(j) = q_j. \quad (49)$$

For example, we can obtain 1-soliton solution to the coupled ANNV equations (35) and (37)

$$f = 1 + \exp(\eta), \quad g_\mu = c_\mu(1) \exp(\eta), \quad (50)$$

$$\eta = px + qy - p^3 t + \eta, \quad \sum_{\mu=1}^M c_\mu(1) = q. \quad (51)$$

Furthermore, we can deduce the following 2-soliton solution:

$$f = -1 - \exp(\eta_1) - \exp(\eta_2) - a_{12} b_{12} \exp(\eta_1 + \eta_2), \quad (52)$$

$$g_\mu = -c_\mu(1) \exp(\eta_1) - c_\mu(2) \exp(\eta_2) + (c_\mu(2) - c_\mu(1)) a_{12} \exp(\eta_1 + \eta_2), \quad (53)$$

$$\sum_{\mu} c_\mu(1) = q_1, \quad \sum_{\mu=1}^M c_\mu(2) = q_2. \quad (54)$$

In the following, we would give the proof of  $N$ -soliton solution to the coupled ANNV equation. First we prove that  $f$  and  $g_\mu$  satisfy the linear equation (37). Expanding  $g_\mu$  with respect to the final character  $\beta_\mu$ , we obtain

$$\begin{aligned} g_\mu &= \text{pf}(d_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_\mu) \\ &= \sum_{j=1}^N \text{pf}(\beta_\mu, b_j) (-1)^{N+j-1} \text{pf}(d_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, \hat{b}_j, \dots, b_N) \\ &= \sum_{j=1}^N c_\mu(j) (-1)^{N+j} \text{pf}(d_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, \hat{b}_j, \dots, b_N), \end{aligned} \quad (55)$$

where  $\hat{\phantom{x}}$  indicates deletion of the character under it. The sum of  $g_\mu$  over  $\mu$  gives

$$\sum_{\mu=1}^M g_\mu = \sum_{j=1}^N q_j (-1)^{N+j} \text{pf}(d_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, \hat{b}_j, \dots, b_N), \quad (56)$$

which is expressed, introducing a new character  $c_0$ , by a Pfaffian

$$\sum_{\mu=1}^M g_\mu = \sum_{j=1}^N q_j (-1)^{N+j} \text{pf}(d_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, \hat{b}_j, \dots, b_N) \quad (57)$$

$$= -\text{pf}(d_0, c_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N), \quad (58)$$

where new entries are defined by

$$\text{pf}(d_0, c_0) = 0, \quad \text{pf}(c_0, a_j) = 0, \quad \text{pf}(c_0, b_j) = q_j. \quad (59)$$

From the deduction in Section 2, we know that

$$\sum_{\mu=1}^M g_\mu = -\text{pf}(d_0, c_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N) = \frac{\partial f}{\partial y}. \quad (60)$$

Next we show that  $f$  and  $g_\mu$  satisfy the bilinear equation (35). The bilinear equation (35) is rewritten as

$$\left( \frac{\partial g_\mu}{\partial t} + \frac{\partial^3 g_\mu}{\partial x^3} \right) f - g_\mu \left( \frac{\partial f}{\partial t} + \frac{\partial^3 f}{\partial x^3} \right) + 3 \left( \frac{\partial g_\mu}{\partial x} \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 g_\mu}{\partial x^2} \frac{\partial f}{\partial x} \right) = 0. \quad (61)$$

Through calculation we have the following differential formulae:

$$f = \text{pf}(d_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_0), \quad (62)$$

$$\frac{\partial f}{\partial x} = -\text{pf}(d_0, d_1, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N), \quad (63)$$

$$\frac{\partial^2 f}{\partial x^2} = -\text{pf}(d_0, d_2, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N), \quad (64)$$

$$\frac{\partial^3 f}{\partial x^3} = -\text{pf}(d_0, d_3, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N) - \text{pf}(d_0, d_1, d_2, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_0), \quad (65)$$

$$\frac{\partial f}{\partial t} = \text{pf}(d_0, d_3, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N) - 2 \text{pf}(d_0, d_1, d_2, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_0), \quad (66)$$

$$g_\mu = \text{pf}(d_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_\mu), \quad (67)$$

$$\frac{\partial g_\mu}{\partial x} = \text{pf}(d_0, d_1, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_\mu, \beta_0), \quad (68)$$

$$\frac{\partial^2 g_\mu}{\partial x^2} = \text{pf}(d_0, d_2, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_\mu, \beta_0), \quad (69)$$

$$\frac{\partial^3 g_\mu}{\partial x^3} = \text{pf}(d_0, d_3, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_\mu, \beta_0) \\ - \text{pf}(d_0, d_1, d_2, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_\mu), \quad (70)$$

$$\frac{\partial g_\mu}{\partial t} = -\text{pf}(d_0, d_3, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_\mu, \beta_0) \\ - 2\text{pf}(d_0, d_1, d_2, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N, \beta_\mu). \quad (71)$$

The detailed calculation is similar as that in [2]. Substituting these relations into Eq. (61), we find that the bilinear equation is reduced to the Pfaffian identity [11],

$$\text{pf}(d_2, \dots) \text{pf}(d_1, \beta_\mu, \beta_0, \dots) - \text{pf}(d_1, \dots) \text{pf}(d_2, \beta_\mu, \beta_0, \dots) - \text{pf}(\beta_\mu, \dots) \text{pf}(d_1, d_2, \beta_0, \dots) \\ + \text{pf}(\beta_0, \dots) \text{pf}(d_1, d_2, \beta_\mu, \dots) = 0, \quad (72)$$

where the list  $\{\dots\}$  represents  $\{d_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N\}$ . So we have proved that  $f$  and  $g_\mu$  satisfy the bilinear equation (35).

#### 4. Bilinear Bäcklund transformation and Lax pair for (22)–(23)

In this section, we will firstly present a bilinear Bäcklund transformation for Eqs. (16)–(17). In fact, concerning Eqs. (16)–(17), we have the following Bäcklund transformation:

$$D_x(g_i \cdot f' - f \cdot g'_i) - \lambda_i D_x f \cdot f' = 0, \quad i = 1, 2, \dots, M, \quad (73)$$

$$(D_t + D_x^3) f \cdot f' = 0, \quad (74)$$

$$(D_t + D_x^3)(g_i \cdot f' + f \cdot g'_i) = 0, \quad i = 1, 2, \dots, M, \quad (75)$$

between Eqs. (16)–(17) and

$$(D_t + D_x^3) g'_j \cdot f' = 0, \quad j = 1, 2, \dots, M, \quad (76)$$

$$D_y^2 f' \cdot f' = 2D_y \left( \sum_{j=1}^M g'_j \right) \cdot f', \quad (77)$$

where we have assumed that

$$g_1 + g_2 + \dots + g_M = f_y, \quad g'_1 + g'_2 + \dots + g'_M = f'_y \quad (78)$$

such that Eqs. (17) and (77) are satisfied automatically and  $\lambda_i$  ( $i = 1, 2, \dots, M$ ) are arbitrary constants. From the Bäcklund transformation we can construct 1-soliton solution from the trivial solution  $f' = 1$ ,  $g'_i = 0$  and we can show the 1-soliton we obtained is the same as that in Section 3. Starting from (73)–(75) with (78), we can derive a Lax pair for (20) and (21). To this end, we set

$$f = \phi f', \quad g_i = \psi_i f' + \phi g'_i, \quad v_i = \frac{2D_x g'_i \cdot f'}{f'^2}, \quad (79)$$

$$\mathbf{c} \cdot \mathbf{v} = \frac{D_x D_y f' \cdot f'}{f'^2}, \quad u = 2(\ln f')_{xx}, \quad \sum_{i=1}^M \lambda_i = \lambda. \quad (80)$$

Then from the Bäcklund transformation (73)–(75), we can deduce that

$$\psi_{i,x} = -v_i \psi_i + \lambda_i \phi, \quad i = 1, 2, \dots, M, \quad (81)$$

$$\psi_{i,t} = (3u v_i + v_{i,xx}) \phi - (3\lambda_i u + v_{i,x}) \phi_x + v_i \phi_{xx} - \lambda_i \phi_{xxx}, \quad (82)$$

$$0 = \phi_t + \phi_{xxx} + 3\phi_x u, \quad (83)$$

$$0 = \phi_{xy} + (\mathbf{c} \cdot \mathbf{v}) \phi - \lambda \phi_x, \quad (84)$$

where  $\mathbf{v} = (v_1, v_2, \dots, v_M)$ ,  $\mathbf{c} = (1, 1, \dots, 1)$  and the inner product  $\mathbf{c} \cdot \mathbf{v} = \sum_{i=1}^M v_i$ . We can check that the compatibility condition of (81)–(84) gives Eqs. (20)–(21).

## 5. Conclusion and discussions

In this paper, we have utilized bilinear formalism to generate a  $(2 + 1)$ -dimensional vector ANNV equation, which is a generalization of the ANNV equation. To the best of our knowledge the vector ANNV equation is new. We hope to find physical applications of Eqs. (20)–(21) in the future. Soliton solutions expressed by Pfaffians have been obtained. Besides, bilinear Bäcklund transformation and the corresponding Lax pair for the vector ANNV equation have been derived. We hope that the bilinear approach can be used to exploit other vector  $(2 + 1)$ -dimensional integrable systems.

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